Flexural Vibrations and Timoshenko's Beam Theory

BIJAN AALAMI*
Arya-Mehr University of Technology, Tehran, Iran
AND
BRUNO ATZORI†
Politecnico di Torino, Italy

This paper is a study of flexural elastic vibrations of Timoshenko beams with due allowance for the effects of rotary inertia and shear. Two independent formulations are developed, one based on the concepts proposed by Timoshenko and the other on the extended Rayleigh-Ritz energy method. The results obtained from the two formulations are matched in order to determine the correcting coefficients in the simplified formulae which are proposed for the frequency of the first two flexural branches. The proposed formulae are shown to achieve greater accuracy in describing the flexural motions. New accurate solutions are offered for several cross-sectional geometries which enable the accuracy of the available methods to be assessed. Correction coefficients are evaluated for several sections and are compared with previous work.

Nomenclature

a = width of rectangular cross section
A = cross-sectional area
b = depth of rectangular cross section
d = a representative length of cross section (Fig. 2)
E = Young's modulus
G = shear modulus
I = moment of inertia of section
$[k]$ = element stiffness matrix (9×9)
$[K]$ = assemblage stiffness matrix
K, K', K'' = correcting coefficients
$[m]$ = element mass matrix
$[M]$ = assemblage gass matrix
M = bending moment
n = constant multiplier of modes
Q = shear force
$\{r\}$ = nodal values of functions w', u', v'
$\{r_o\}$ = nodal amplitudes of functions w', u', v'
r = radius of gyration
t = time
t_f = flange thickness
t_w = web thickness
T = kinetic energy
u, v, w = displacements in x, y, and z directions, respectively
u', v', w' = displacement functions in terms of x, y, and t
V = potential energy
x, y, z = Cartesian coordinates
y_{max} = amplitude of vibrations
α = time dependent coefficients
γ = rotation of centroidal axis due to shear
θ = angle describing cross sectional geometry
ε = normal strains
λ = half wave length
ν = Poisson's ratio
ξ = wave length ratio ($=r/\lambda$)
ρ = density
σ = stress
ω = circular frequency
Ω = normalized circular frequency $[=(\omega r/\pi)(\rho/G)^{1/2}]$
$(\dot{\ })$ = differentiation with respect to time

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* Associate Professor of Structural Mechanics.

† Assistant Professor of Machine Design.

Introduction

THERE has been considerable research interest in the analysis of flexural motions of elastic beams for which the effects of cross-sectional dimensions on frequencies cannot be neglected. The classical one-dimensional Bernoulli-Euler theory of flexural vibrations is known to be inadequate for the aforementioned beams and for higher modes. Corrections due to rotatory inertia and shear in the classical theory were first introduced by Timoshenko.[24,25] Beams analyzed with such refined theories are referred to as Timoshenko beams.

Although much work has been proposed for the vibration analysis of Timoshenko beams, no clear-cut and generally acceptable resolution of the problem has been formulated thus far. Due to a general paucity of exact solutions in this field, difficulties have been encountered in making effective simplifying assumptions in the approximate approaches. Moreover, it has not been possible to assess the degree of approximation of the analyses proposed.

If the wave length of the transverse vibrations is large in comparison with the dimensions of the cross section, such as in the case of large structural elements, the correction to classical theory may be less significant.

However, the correction is the essence of the questions which arose in the design of vibrating elements of certain electronics circuit components, such as acoustic wave guides. Furthermore, it is important in higher beam modes and branches of vibrations such as recurs in aerospace applications. Beside the first branch, the application of Timoshenko beam theory to transient responses of beams requires the accurate description of higher branches; for such application the theory in its present standing is clearly inadequate.

There is a comprehensive survey of the extensive work done in this field in a recent work of Cowper.[8,9] The basic approach, as proposed by Timoshenko[24,25] is the addition of rotatory inertia and shear terms to the classical equations of motion. For the latter term, a correction factor is considered. Most of the exisung work is concerned with the determination of this coefficient for various cross sections. Cowper[8] adopted a somewhat different approach of redefining the value of K (correcting coefficient) while keeping the concept of a single correction factor for each section. More work on this subject is reported in Refs. 4, 7, 10–12, 15, and 17.

Herein a three-dimensional formulation based on the extended Rayleigh-Ritz energy method and developed for the vibration analysis of prismatic bars is briefly outlined (Formulation 2). Based on this formulation new exact solutions are obtained for several general cross sections enabling the accuracy of the proposed approximate methods to be checked. The solutions are termed exact, in that they account for the three-dimensional state of stress and the cross-sectional geometry. In light of the solutions obtained, the Timoshenko approach and those of the subsequent investigators are critically reviewed. It is concluded that the flexural frequencies of Timoshenko beams cannot be expressed satisfactorily with only one correcting coefficient. Thus modified formulae (Formulation 1) are proposed with significant improvements on the accuracy. The differences among the proposed formulae and previous works are pointed out. New values of correcting coefficients are obtained and compared with the available values in the literature.

Analysis

The analysis is comprised of two independent approaches: Formulation 1 based on the concepts attributed to Timoshenko, and Formulation 2 based on a three-dimensional form of extended Rayleigh-Ritz energy method. The results of the two formulations are matched to determine certain correcting coefficients in the simple formulae proposed for the frequency equations. It should be noted here that both formulations refer to simply supported beams.

For the purpose of this paper, first branch refers to the lowest frequency of the flexural motion of a beam. In its first mode, this motion is one half wave over the span, and its higher modes are associated with n number of half waves over same span. The second branch refers to the next higher frequency form of flexural vibrations of a beam, which also consists of one half wave length over its span in its first mode.

Formulation 1

Consider the equilibrium of an element dz of an elastic isotropic beam under flexural vibrations (z-direction along the beam, y in direction of transverse vibrations). The angle γ is rotation due to shear of the tangent of the bent axis of the beam and the angle ϕ is rotation of the same tangent due to bending moment M. The equation of motion for the rotation of the element is

$$-(\partial M/\partial z)\cdot dz+Qdz = \rho I(\partial^2\phi/\partial t^2)\cdot dz \tag{1}$$

It has long been established that the deformations of the centroidal axis of a Timoshenko beam together with the assumption of linear distribution of axial strain over the cross section are not sufficient to express its complex vibrational behavior satisfactorily. Timoshenko[24] proposed a constant correction factor (K') for the shear angle γ. Herein an additional correction factor K'' is introduced for the curvature of the centroid to allow for nonlinear distribution of direct strain over the cross section. Thus

$$Q = K'GA\gamma \tag{2}$$

and

$$M = -K''EI(\partial\phi/\partial z) \tag{3}$$

Coefficients K' and K'' depend on the geometry of the cross section and are assumed to be constant for all frequencies. Leonard and Budiansky[19] used a coefficient similar to K'' for the analysis of transient vibrations of bars.

Using the geometry of deformation

$$Q = K'GA[(\partial y/\partial z)-\phi] \tag{4}$$

Substitution for M and Q in Eq. (1) yields the rotational equation of motion of the element which simplifies to

$$EIK''\frac{\partial^2\phi}{\partial z^2} + GAK'\left(\frac{\partial y}{\partial z} - \phi\right) - \rho I\frac{\partial^2\phi}{\partial t^2} = 0 \tag{5}$$

Now for translation in the y direction

$$\partial Q/\partial z\cdot dz = \rho A(\partial^2 y/\partial t^2)\cdot dz \tag{6}$$

which on substitution for Q from Eq. (2) results in the translational equation of motion of the element in terms of ϕ, x, and y

$$GAK'\left(\frac{\partial^2 y}{\partial z^2} - \frac{\partial\phi}{\partial z}\right) - \rho A\frac{\partial^2 y}{\partial t^2} = 0 \tag{7}$$

Eliminating ϕ from Eqs. (5) and (7) results in a single equation of motion as follows:

$$EIK''\frac{\partial^4 y}{\partial z^4} + \rho A\frac{\partial^2 y}{\partial t^2} - \rho A\left(1 + \frac{EK''}{GK'}\right)\frac{\partial^4 y}{\partial z^2\,\partial t^2} + \frac{\rho^2 I}{GK'}\frac{\partial^4 y}{\partial t^4} = 0 \tag{8}$$

For a simple harmonic motion, a solution of the following type may be assumed

$$y = y_{\max}\sin(n\pi z/\lambda)\cos\omega t \tag{9}$$

in which λ is the half wave length.

On substitution of the assumed y displacements in the equation of motion (8) and simplifications, the frequency equation is obtained (also substitute r^2 for I/A)

$$\frac{\rho r^2}{GK'}\omega^4 - \left[1 + \frac{r^2 n^2\pi^2}{\lambda^2}\left(1 + \frac{EK''}{GK'}\right)\right]\omega^2 + \frac{Er^2K''n^4\pi^4}{\rho\lambda^4} = 0 \tag{10}$$

The solutions of the frequency Eq. (10) for the first mode are

$$\omega = \frac{1}{r}\left(\frac{GK'}{2\rho}\right)^{1/2}\left\{\left[1 + \frac{r^2\pi^2}{\lambda^2}\left(1 + \frac{EK''}{GK'}\right)\right] \pm \left[\left[1 + \frac{r^2\pi^2}{\lambda^2}\left(1 + \frac{EK''}{GK'}\right)\right]^2 - \frac{4\pi^4 r^4 EK''}{G\lambda^4 K'}\right]^{1/2}\right\}^{1/2} \tag{11}$$

For the purpose of the present analysis, using $\xi = r/\lambda$ and normalized frequency $\Omega = \omega r/\pi(\rho/G)^{1/2}$, the frequency expression may be normalized in the following form:

$$\Omega = \xi\left(\frac{E}{2G}\right)^{1/2}\left\{\left[K'' + \frac{GK'}{E}\left(1 + \frac{1}{\pi^2\xi^2}\right)\right] \pm \left[\left[K'' + \frac{GK'}{E}\left(1 + \frac{1}{\pi^2\xi^2}\right)\right]^2 - \frac{4GK'K''}{E}\right]^{1/2}\right\}^{1/2} \tag{12}$$

The frequency expression (12) refers to the first flexural branch with the negative sign and the second flexural branch with the positive sign. Thus far, most of the previous investigators have disregarded the solution with the positive sign concerning themselves with the expression of the first branch for the study of the dispersion characteristics of Timoshenko beams. Nonetheless, in some works, such as Ref. 14, the formulation is applied to the second branch in that the cutoff frequency of this branch is used to evaluate a shear correction coefficient for the section under consideration.

From the preceding formulation, provided the values of K' and K'' are known, the dispersion spectrum of a beam may be readily evaluated for both branches from Eq. (12).

It is of interest to note the relation of the preceding part of this formulation with the previous work.

The frequency equation (8) reduces to that used by Timoshenko[24] with $K'' = 1$. In Ref. 9, however, an approximate solution is obtained for the first branch and for small values of ξ. Goodman[14] begins with the same type of frequency equation (8) and leads to an accurate solution for Ω for the first branch with $K'' = 1$. Anderson[5] arrives at both solutions of Ω, similar to the one expressed in Eq. (12), but with $K'' = 1$. Cowper[9] introduces a new approach for the definition of the correcting coefficient K, but concludes with a similar expression to Anderson[5] with one unknown coefficient for all wave lengths of each section.

In the work of Leonard and Budiansky[19] concerning the analysis of the problem of travelling waves in bars, an expression is derived for the frequency spectrum which contains both branches and makes allowance for the Timoshenko beam inaccuracies by means of two constant coefficients comparable to K'' and K'.

One of the major differences among the various available approaches, as well as the analysis presented herein, is the method of evaluating the correcting coefficients K. Timoshenko[24] has based the evaluation on the distribution of shearing stresses over the cross section. Several investigators matched the wave velocity obtained from the $\xi \to \infty$ limit of frequency expressions of types of Eq. (8) to that of Rayleigh waves: the matching of the two velocities yields the value of one unknown K. Another approach used[14] is to determine the cutoff frequency of the second branch given by the frequency Eq. (12) and equate it to the known circular frequency of thickness-shear mode for thin rectangular bars. The value of K thus obtained has been used to describe the motion of the first branch. Finally one other method has been to match the frequency values given from expressions similar to Eq. (12) at selective points with the exact solutions whenever available. So far this procedure has been greatly handicapped due to the general paucity of exact solutions.

A study of the results of the exact analysis for several cross sections and their comparison with the available approximate solutions have lead to the conclusion that it is not feasible to determine a single general constant value of K for the accurate description of the vibration behavior of a beam over a wide range of wavelengths. For a reasonably good agreement between the approximate and exact solutions, the values of K for the two branches generally should be assumed different. Thus, K_1'' and K_1' may be taken to refer to the frequently discussed first flexural branch with K_2'' and K_2' describing the second flexural branch. Based on this postulation, the formulation may be extended as follows into simple formulae with significant improvements in the capability of describing the two branches accurately.

2-K formula

Corresponding with the assumptions of the previous investigators, K'' is assumed to be equal to 1. The coefficients K_1' and K_2' are used to describe the first and second branches, respectively, as given by Eq. (13) (minus for first and plus for second branch)

$$\Omega_1 = \xi\left(\frac{E}{2G}\right)^{1/2}\left\{\left[1+\frac{GK_1'}{E}\left(1+\frac{1}{\pi^2\xi^2}\right)\right] \mp \left[\left[1+\frac{GK_1'}{E}\left(1+\frac{1}{\pi^2\xi^2}\right)\right]^2 - \frac{4GK_1'}{E}\right]^{1/2}\right\}^{1/2} \tag{13}$$

The values of these coefficients are determined by matching the frequency given by the Eqs. (13) at selected points with those of the exact analysis of Formulation 2. For the first branch, matching at $\xi = 1$, as will be discussed later, gives a reasonably good description of frequency spectrum for the range of infinitely long wavelengths to wavelengths of the same order of magnitude as the relating radius of gyration of the section. For the second branch, K_2' is evaluated by matching the frequencies at cutoff.

4-K formula

This formula is expressed by Eq. (14), for which values of K_1'', K_1' for the first branch and K_2'', K_2' for the second branch should be evaluated (minus for first and plus for second branch)

$$\Omega_1 = \xi\left(\frac{E}{2G}\right)^{1/2}\left\{\left[K_1''+\frac{GK_1'}{E}\left(1+\frac{1}{\pi^2\xi^2}\right)\right] \mp \left[\left[K_1''+\frac{GK_1'}{E}\left(1+\frac{1}{\pi^2\xi^2}\right)\right]^2 - \frac{4GK_1'K_1''}{E}\right]^{1/2}\right\}^{1/2} \tag{14}$$

The coefficients K are determined following the same procedure as outlined for the 2-K Formula except that two points for each branch are considered for matching giving a closer agreement between the simple formulas of Eq. (14) and the exact solutions. These formulas are found to be a major improvement to the 2-K Formula for the second branch.

Formulation 2

For the analysis of steady-state vibrations of elastic isotropic bars, the displacements on a cross section parallel to x-y plane, and at time t, using the finite element approach, may be taken as follows:

$$\begin{aligned} w &= w'(x,y,t)\sin(\pi z/\lambda) \\ u &= u'(x,y,t)\cos(\pi z/\lambda) \\ v &= v'(x,y,t)\cos(\pi z/\lambda) \end{aligned} \tag{15}$$

The functions w', u', and v' may be resolved into time dependent generalized coordinates α and position dependent Cartesian coordinates in the following form:

$$\begin{aligned} w' &= \alpha_1+\alpha_2 x+\alpha_3 y \\ u' &= \alpha_4+\alpha_5 x+\alpha_6 y \\ v' &= \alpha_7+\alpha_8 x+\alpha_9 y \end{aligned} \tag{16}$$

From the diplacement functions of Eqs. (16) the strain energy of an element of volume $dx\,dy\,dz$ may be evaluated using Eq. (17)

$$\text{S.E.} = \tfrac{1}{2}\int\int\int \{\varepsilon\}^T\{\sigma\}\,dx\,dy\,dz \tag{17}$$

Where for any given wave length ratio $\xi(=r/\lambda)$, the strains $\{\varepsilon\}$ at a point are derived by appropriate differentiation of displacement functions. Stresses $\{\sigma\}$ are subsequently expressed in terms of strains using the generalized Hooke's law. Similarly for an element of volume $dx\,dy\,dz$, with uniform mass density ρ, the kinetic energy is given by

$$\text{K.E.} = \frac{1}{2}\rho\int\int\int[\dot{w}\,\dot{u}\,\dot{v}]\begin{Bmatrix}\dot{w}\\ \dot{u}\\ \dot{v}\end{Bmatrix}dx\,dy\,dz \tag{18}$$

in which the velocities $\dot{w}$, $\dot{u}$, and $\dot{v}$ are derived through differentiation of Eq. (16) with respect to time t.

The beam is considered being mathematically subdivided into an assemblage of discrete segments (elements) in form of triangular prisms with their longitudinal axis parallel to the axis of the beam.

The potential energy (V) of each element, being equal to its strain energy in this case, can be evaluated by integration of Eq. (17), over the volume of the element for the representative length λ and be expressed in form of Eq. (19), where $\{r\}$ is the nodal values of functions w', u', and v' from Eq. (16), and $[k]$ is the stiffness matrix of the element

$$V = \tfrac{1}{2}\{r\}^T[k]\{r\} \tag{19}$$

Similarly the representative kinetic energy (T) of an element of length λ may be evaluated by integration of Eq. (18) over the same volume and simplified into Eq. (20)

$$T = \tfrac{1}{2}\{\dot{r}\}^T[m]\{\dot{r}\} \tag{20}$$

The Lagrangian L of a representative length λ of the bar is given by the summation over the cross section of the appropriate energies of each element

$$L = \sum(T-V) = \frac{1}{2}\{\dot{r}\}^T[M]\{\dot{r}\} - \frac{1}{2}\{r\}^T[K]\{r\} \tag{21}$$

where $[M]$ and $[K]$ are, respectively, the mass and stiffness matrices of the assemblage formed through summation of appropriate terms of elements' stiffness and mass matrices.

Application of Hamilton's principle to the Lagrangian of Eq. (21) gives the following equations of motion[16]

$$[K]\{r\}+[M]\{\ddot{r}\} = 0 \tag{22}$$

Table 1 Homogeneous isotropic solid cylinder. Accuracy comparison of the five lowest frequencies; frequency $\Omega = \omega/\omega_s$, where $\omega_s^2 = (\pi/l)^2 G/\rho$[a]

Mode	$1/\lambda = 1$			
	AGH[b]	Present analysis	% Difference	Description
1	0.8766	0.8819	0.6	Flexure 1st mode
2	1.0000	1.0000	0.0	Torsion 1st mode
3	1.0607	1.0832	2.2	$n = 2$[c] 1st mode
4	1.1096	1.1290	1.8	Longitudinal 1st mode
5	1.2618	1.2861	2.1	Flexure 2nd mode

[a] l = radius of cylinder.
[b] Armenakas, Gazis, and Herrmann.[9]
[c] n = number of waves along the circumference.

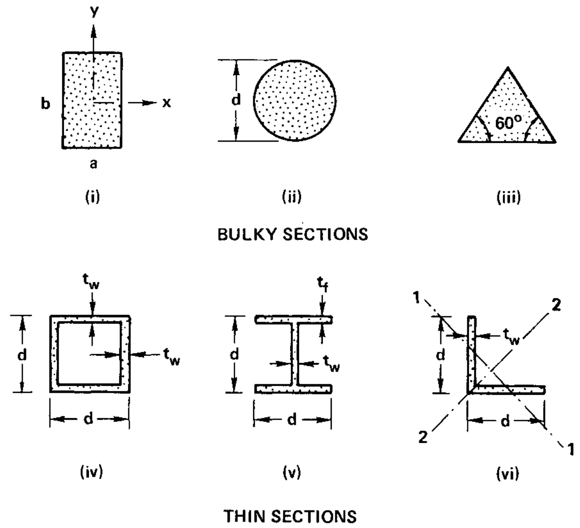

Fig. 1 Cross-sectional geometry of prismatic bars analyzed.

For simple harmonic motion with $\{r_o\}$ being the amplitudes of nodal values of functions w', u', and v', Eq. (23) may be written

$$\{r\} = \{r_o\} e^{i\omega t} \tag{23}$$

where ω is the circular frequency. Subsequently on substitution for $\{r\}$ from above into equations of motion (22), the analysis of the vibrations of the beam reduces to the following algebraic eigenvalue problem:

$$([K] - \omega^2[M])\{r_o\} = 0 \tag{24}$$

An efficient iterative method as outlined in Ref. 13 is employed to evaluate a predetermined number of lowest eigenvalues and their eigenvectors without a priori knowledge of mode configurations or imposition of mode restraints. The effectiveness of this general method of analysis and its scope are discussed in Refs. 1–3. The accuracy is demonstrated in Table 1 in which the results of the present analysis for a homogeneous solid cylinder (subdivided into 96 elements) are compared with the exact solutions available.[6]

Discussion and Results

Beside the rectangular and the circular cross sections, solutions have not been available for the vibrations of the first and the second flexural branches of prismatic bars of arbitrary cross sections. Herein a number of typical bulky and thin cross sections are considered as shown in Fig. 1.

Table 2 Homogeneous isotropic solid rectangular bars. Frequencies of the first two flexural branches of vibrations for different wavelengths [frequency ratio $\Omega_r = \omega r/\pi(\rho/G)^{1/2}$, $\nu = 0.3$]

Wave-length ratio r/λ	Ω—First flexural branch				Ω—Second flexural branch		
	Aspect ratio b/a				Aspect ratio b/a		
	6	3	1	1/3	6	3	1
0.001	0.0000	0.0000	0.0000	0.0000	0.2891	0.2896	0.2905
0.01	0.0005	0.0005	0.0005	0.0005	0.2897	0.2900	0.2910
0.1	0.0432	0.0433	0.0434	0.0440	0.3379	0.3381	0.3368
0.2	0.1316	0.1317	0.1321	0.1341	0.4407	0.4409	0.4151
0.4	0.3302	0.3306	0.3318	0.3363	0.6692	0.6700	0.5180
0.6	0.5280	0.5290	0.5316	0.5388	0.8377	0.8425	0.6541
0.8	0.7225	0.7243	0.7296	0.7399	0.9779	0.9828	0.8180
1.0	0.9146	0.9179	0.9272	0.9407	1.1314	1.1355	0.9971

Table 3a Long wavelength frequencies of the first bending mode of bulky sections, $\Omega = (\omega r/\pi)(\rho/G)^{1/2}$ (bending about $x-x$ axis)

r/λ	Cross section				
	Square	Rectangle $b/a = 6$	Triangle (equilateral)	Circle: Present analysis	Circle: Exact[a]
0.001	5.082×10^{-6}	5.067×10^{-6}	5.080×10^{-6}	5.060×10^{-6}	...
0.01	5.073×10^{-4}	5.059×10^{-4}	5.070×10^{-4}	5.050×10^{-4}	...
0.1	4.342×10^{-2}	4.327×10^{-2}	4.271×10^{-2}	4.352×10^{-2}	4.352×10^{-2}
0.2	1.321×10^{-1}	1.316×10^{-1}	1.270×10^{-1}	1.335×10^{-1}	1.333×10^{-2}

[a] Ref. 6.

Table 3b Long wavelength frequencies of the first bending mode of thin cross sections, $\Omega = \omega r/\pi(\rho/G)^{1/2}$

h/λ	Cross section				
	I-section (1)[a]	I-section (2)[b]	Square hollow section (3)[c]	Square hollow section (4)[d]	Angle section (5)[e]
0.002	8.562×10^{-6}	8.520×10^{-6}	7.499×10^{-6}	7.876×10^{-6}	4.005×10^{-6}
0.02	8.519×10^{-4}	8.492×10^{-4}	7.487×10^{-4}	7.861×10^{-4}	4.002×10^{-4}
0.2	6.031×10^{-2}	6.549×10^{-2}	6.506×10^{-2}	6.681×10^{-2}	3.764×10^{-2}
0.4	1.490×10^{-1}	1.620×10^{-1}	1.970×10^{-1}	1.910×10^{-1}	1.128×10^{-1}

[a] Fig. 1—v, $t_f/d = 1/10$, $t_w/d = 1/20$, bending about $x-x$, $r/d = 0.4223$.
[b] Fig. 1—v, $t_f/d = 1/20$, $t_w/d = 1/20$, bending about $x-x$, $r/d = 0.4204$.
[c] Fig. 1—iv, $t_w/d = 1/10$, bending about $x-x$, $r/d = 0.3697$.
[d] Fig. 1—iv, $t_w/d = 1/20$, bending about $x-x$, $r/d = 0.3884$.
[e] Fig. 1—vi, $t_w/d = 1/10$, bending about $1-1$, $r/d = 0.2889$.

Frequency results of a number of common cross-sections obtained from the finite element analysis are given in Fig. 2. The figure illustrates that the vibration characteristics of the sections cannot satisfactorily be described in terms of their radius of gyrations, cross-sectional area and one value of K as proposed by the previous investigators.

Usually, the effect of Poisson's ratio has been neglected for a rectangular section with an uncertain loss of accuracy. The previous investigators have emphasized the use of their analysis

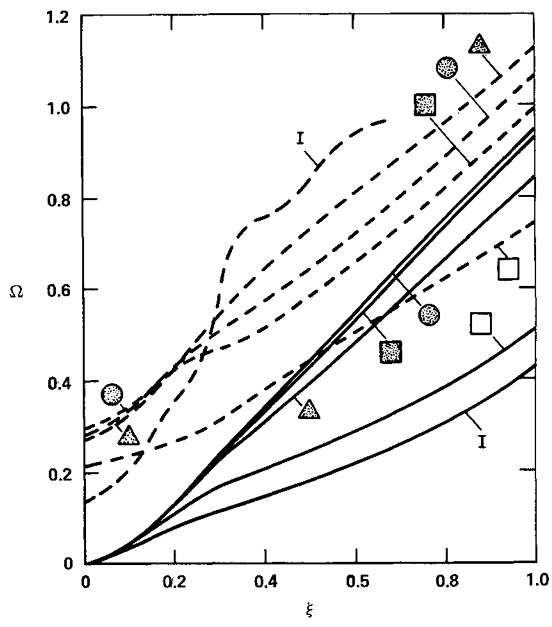

Fig. 2 Frequency spectrum of the lowest two flexural (about $x-x$) branches of homogeneous prismatic bars of Fig. 1 [frequency $\Omega = \omega r/\pi(\rho/G)^{1/2}$, wave length ratio $\xi = r/\lambda$, full line for first branch, broken line for second branch, for I-section $t_w/d = 1/20$, $t_f/d = 1/10$, for square hollow section $t_w/d = 1/10$].

Table 4 Cutoff frequencies of the second flexural mode, $\Omega = \omega r/\pi(\rho/G)^{1/2}$

Section			Second flexural cutoff frequencies Ω
Rectangle	b/a	6	0.2891
		3	0.2896
		1	0.2905
Circle			0.2969
Equilateral triangle			0.2737
Hollow square	t_f/d	1/10	0.2139
		1/20	0.1778
I-Section $t_w/d = 1/20$	t_f/d	1/10	0.1391
		1/20	0.1749
Equal Angle[a]			0.1581

[a] The quoted value is not the lowest cutoff for this section.

for thin sections, for which the loss of accuracy is considered to be negligible. Herein, solutions are presented (Table 2) for rectangular cross sections with aspect ratios $b/a = 6$ to $\frac{1}{3}$ covering the range of thin to wide bars. It is concluded that the approximation involved in a thin bar assumption is very small both for the first branch and for long wave lengths in the second branch.

Table 3 contains the long wavelength frequencies of the first flexural branches of both the thin and the thick sections over a range of interest to structural and mechanical vibrations. The flexural cutoff frequencies of the sections analyzed are given separately in Table 4.

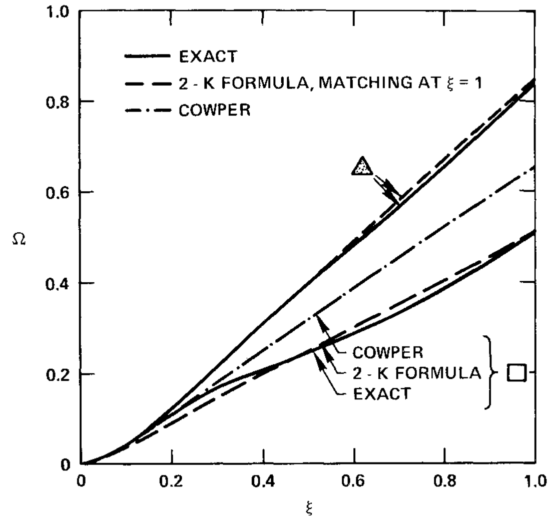

Fig. 4a Homogeneous square hollow section ($t_w/d = 1/10$) and equilateral solid triangle. Comparison of the frequency spectrums of the first flexural mode between the exact solutions, 2-K formula and results based on Cowper's[8] value of K [$\Omega = \omega r/\pi(\rho/G)^{1/2}$, $\xi = r/\lambda$].

For a thin rectangular bar the dispersion curves from the finite element (termed exact) analysis, those of Cowper[8] and the 2-K formula of the present analysis are compared in Fig. 3. Apart from Timoshenko's value of K[24] (not used in the figure) the 2-K formula and Ks proposed by other investigators result in very close behavior. The agreement for the first branch is good. In this case the value of K from the 2-K formula is not very sensitive to the point at which the solutions are matched. However, for the second branch the approximate solutions are grossly in error over the short wavelength range.

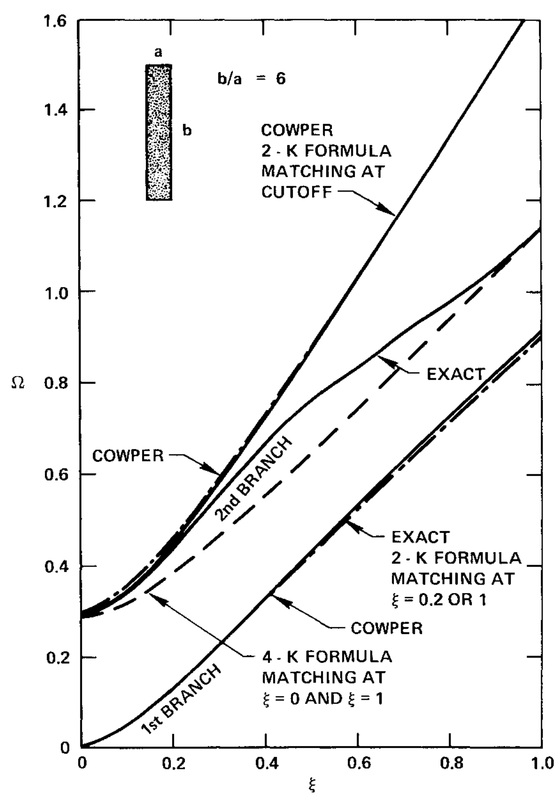

Fig. 3 Frequency spectrum of the lowest two flexural branches of a rectangular bar ($b/a = 6$). Comparison between the exact solutions, 2-K formula, 4-K formula and results based on Cowper's[8] value of K [$\Omega = \omega r/\pi(\rho/G)^{1/2}$, $\xi = r/\lambda$].

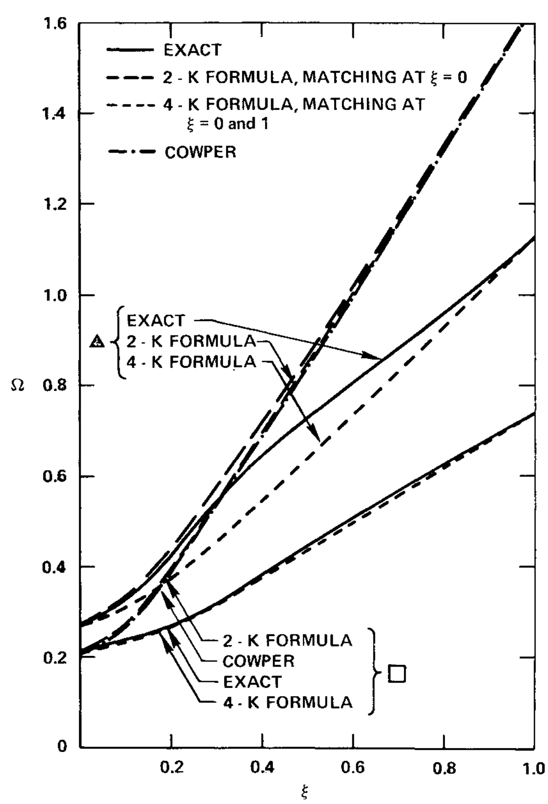

Fig. 4b Homogeneous square hollow section ($t_w/d = 1/10$) and equilateral solid triangle. Comparison of the frequency spectrums of the second flexural mode between the exact solutions, 2-K and 4-K formulae, and results based on Cowper's[8] value of K [$\Omega = \omega r/\pi(\rho/G)^{1/2}$, $\xi = r/\lambda$].

Table 5 Values of coefficient K' based on 2-K formula together with results of other investigators

Section				Values of coefficient K': First branch, matching at $\xi = 0.2$	First branch, matching at $\xi = 1$	Second branch, matching at $\xi = 0$	Other investigators
Bulky sections	Rectangle	$\frac{b}{a}$	6	0.878	0.879	0.823	0.667^a
			3	0.880	0.886	0.827	0.850^b 0.822^c
			1	0.893	0.905	0.832	0.870^d
			1/3	0.957	0.934	...	0.833^e
	Circle			0.937	0.945	0.869	0.750^a 0.886^b 0.847^c 0.900^e
	Equilateral triangle			0.754	0.739	0.739	...
Thin sections	Hollow square	$\frac{t_f}{d}$	1/10	0.414	0.257	0.451	0.435^b
			1/20	0.284	0.096	0.312	
	I-section $t_w/d = 1/20$	$\frac{t_f}{d}$	1/10	0.156	0.184	0.191	0.188^b
			1/20	0.152	0.078	0.302	0.295^b

[a] Timoshenko.[24]
[b] Cowper.[8]
[c] Mindlin.[20]
[d] Goodman.[14]
[e] Roark.[23]

Table 6 Thin rectangular cross section $b/a = 6$. Values of coefficients K' and K'' based on 4-K formula

		Values of K for matching at $\xi = \infty$ and at $\xi = 0.2$	$\xi = 0.5$	$\xi = 1$
First Branch	K_1'	0.86	0.86	0.86
	K_1''	1.02	1.10	1.52
		Values of K for matching at $\xi = 0$ and at $\xi = 0.2$	**$\xi = 0.5$**	**$\xi = 1$**
Second Branch	K_2'	0.823	0.823	0.823
	K_2''	0.9	0.695	0.402

In Figure 4a the error in dispersion curves based on Cowper[8] approach is clearly indicated. For the first branch the accuracy of the 2-K formula is especially good over the range at which the frequencies are matched. Figure 4b illustrates the lack of accuracy of the previous investigators' procedures over the short wave lengths of the second branch.

Table 5 lists the values of the K obtained from the 2-K approximate formula for the sections considered. Two values of K are obtained for the first branch, one matching at $\xi = 0.2$ for long wavelength range, and the other at $\xi = 1$ for short wavelengths. Values proposed by the other investigators are also included for comparison when available.

Based on the 4-K formula, values of K are evaluated for the rectangular cross section with $b/a = 6$ matching at four points as given in Table 6. For this particular section, the previous approximate solutions are good for the first branch, but for the second branch significant improvement is achieved through the use of this formula as shown in Fig. 3.

Similar solutions for the triangular section and one of the thin sections are obtained and the resulting dispersion curves are plotted in Fig. 4b for the second branch.

Conclusions

A study of the results of the analysis and the discussions indicate the following.

a) For the rectangular cross section, the error involved in the use of the approximate thin section formula is very small for the first flexural branch. For the second flexural branch the error is larger and approximates 10% over the short wavelength range covered.

b) The values of K proposed by previous investigators result in good agreement with theory only for the first flexural branch of rectangular cross sections.

c) For thin sections Cowper values of K are the only available solutions. These can be substantially improved by the 2-K formula in the first flexural branch, and through the use of the 4-K formula both in the first and the second branches.

d) The proposed 2-K formula improves the dispersion curve accuracy of both the thin and the bulky sections for the first flexural branch. For the second branch, however, it is in error of the same order of magnitude as the other methods available so far.

e) The 4-K formula gives accurate results for all sections and for both branches.

f) The use of the 2-K formula for the first branch and the 4-K formula for the second branch offers an optimum for keeping the number of Ks to a minimum (three in this case), and yet being able to express the dispersion characteristics of a bar accurately.

g) The present work should be extended to evaluate the values of K based on the proposed 4-K formula for all the sections analyzed. Other sections of interest may also be analyzed and their corresponding values of K evaluated.

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Nonlinear Panel Response by a Monte Carlo Approach

R. Vaicaitis*
Columbia University, New York, N.Y.

AND

E. H. Dowell† and C. S. Ventres‡
Princeton University, Princeton, N.J.

The vibration of clamped and simply supported elastic panels due to subsonic and supersonic turbulent boundary-layer flows is investigated by a Monte Carlo technique. The resulting generalized random forces are simulated numerically from boundary-layer turbulence spectra and the response analysis is performed in time domain. The mutual interaction between panel motion and external and/or internal airflow is included. Response studies are performed with respect to rms response, probability structure, peak distribution, threshold crossing and spectral density. The effect on the response statistics of in-plane loading, static pressure differential and cavity pressure is investigated.

Nomenclature

a = plate length
a_∞ = velocity of sound, external flow
a_c = velocity of sound, cavity flow
b = plate width
b_{ij} = nondimensional modal amplitude
D = $Eh^3/12(1-\nu^2)$, plate stiffness
d = cavity depth
E = modulus of elasticity
i = imaginary unit $(-1)^{1/2}$
h = plate thickness
k_1, k_2 = wave numbers in x, y directions, respectively
M = Mach number, u_∞/a_∞
r, n = mode numbers
q = $\frac{1}{2}\rho_\infty u_\infty{}^2$, dynamic pressure
t = time
T = period
u_c = convection velocity
u_∞ = mean external flow velocity
w = plate deflections
x, y, z = spatial coordinates
ζ = boundary-layer displacement thickness
ζ^* = boundary-layer thickness
η = y/b
λ = $\rho_\infty u_\infty{}^2 a^3/2D$
λ^c = $\rho_c a_c{}^2 a^3/D$
μ = $\rho_\infty a/\rho_m h$
μ^c = $\rho_c a/\rho_m h$
ν = Poisson's ratio
ξ = x/a
ρ_m = plate material density/unit area
ρ_∞ = freestream density
ρ_c = cavity flow density
τ = $t(D/\rho_m h a^4)^{1/2}$
ω = frequency

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Index categories: Structural Dynamic Analysis; Aeroelasticity and Hydroelasticity.
* Assistant Professor of Civil Engineering. Member AIAA.
† Professor, Department of Aerospace and Mechanical Sciences. Member AIAA.
‡ Research Staff, Department of Aerospace and Mechanical Sciences.

1. Introduction

In Refs. 1–5, parametric studies were performed on nonlinear panel response and panel flutter using a time domain analysis. In Ref. 3 the boundary-layer turbulence was limited to gaussian white noise while in Ref. 5 the analysis considered only two streamwise modes. Simply supported boundary conditions were assumed in all of these cases. This paper is mainly concerned in studying statistical properties of nonlinear panel response to more realistic models of boundary layer turbulence and extending the analysis to clamped support boundary conditions.

A Monte Carlo technique is employed for the response analysis of panels undergoing large deformations under subsonic and